

# Example of Hassenpflug Matrix Tensor Notation

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## Abstract

This is a sample document to illustrate the typesetting of vectors, matrices and tensors according to the matrix tensor notation of Hassenpflug (1993, 1995). The first section describes the bare basics of the notation and please note that there is much more to the notation than the little bit described here.

**Keywords:** vector, matrix, tensor, notation.

***N.B.:** This document is neither a guide nor a reference document for the Hassenpflug notation. For any reference to the material in section §1.1, please cite the original copyrighted articles (Hassenpflug 1993, 1995).*

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# 1 Matrix tensor notation

## 1.1 Basic vector notation

All vectors are in the 3-dimensional Euclidean space  $\mathbb{R}^3$  and tensors in  $\mathbb{R}^{3 \times 3}$ . Any other vector space will be explicitly stated. The rest of this section lists the basic definitions of the notation of Hassenpflug (1993, 1995)

$$\text{Physical vector:} \quad \vec{x} \equiv \vec{e}_1 x_1 + \vec{e}_2 x_2 + \vec{e}_3 x_3 \quad (1.1)$$

The physical vector is the general representation of a vector in any coordinate system. The unit vectors  $\vec{e}_i$ , ( $i = 1, 2, 3$ ), define the direction of the axes in a right-handed orthogonal Cartesian system. The components,  $\vec{e}_i x_i$ , are the components of the vector and the scalar quantities,  $x_i$ , the elements of the vector.

$$\text{Column vector:} \quad \bar{x}^a \equiv \begin{bmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \end{bmatrix} \quad (1.2)$$

The column matrix of the elements of a vector is called a column vector and is the algebraic representation of a vector. The bar above the symbol of the vector indicates a column vector and the superscript ( $a$ ) the index of the specific coordinate system in which the elements of the vector are expressed.

$$\text{Row vector:} \quad \underline{x}_a \equiv [\bar{x}^a]^T = [x_{a1} \quad x_{a2} \quad x_{a3}] \quad (1.3)$$

The row matrix of the elements of a vector is called a row vector. The bar below the symbol of the vector indicates a row vector and the subscript ( $a$ ) the index of the specific coordinate system in which the elements of the vector are expressed. It is important to note that in general is  $[\bar{x}^a]^T = \underline{x}_a^T$  for skew and curved coordinates (see Hassenpflug 1995). The format in equation (1.3) without the transpose sign is only valid in Cartesian coordinates.

$$\text{Norm:} \quad \|\vec{x}\| \equiv x, \quad (1.4a)$$

$$\|\bar{x}\| \equiv x \equiv \sqrt{\underline{x} \cdot \bar{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad (1.4b)$$

The norm of a vector is the algebraic size or length of the vector. The second equation, (1.4b), in element form, is only valid in Cartesian coordinates or Euclidean space.

$$\text{Scalar, dot or inner product:} \quad \vec{x} \bullet \vec{y} \equiv \underline{x} \cdot \bar{y} = x y \cos \varphi, \quad (1.5a)$$

$$\bar{x} \bullet \bar{y} \equiv \underline{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 \quad (1.5b)$$

The scalar product of two vectors results in a scalar. The angle  $\varphi$  is the angle in space between  $\vec{x}$  and  $\vec{y}$ .

$$\text{Dyad or outer product:} \quad \bar{x} \circ \bar{y} \equiv \bar{x} \cdot \underline{y} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{bmatrix} \quad (1.6)$$

The dyad or outer product of two vectors results in a square matrix. There exists a well-defined algebra for dyads. It is sometimes convenient to handle second-rank Cartesian tensors such as inertia tensors as a linear polynomial of dyads, called a dyadic.

$$\begin{aligned} \text{Vector or cross product:} \quad \vec{x} \times \vec{y} &\equiv (x_2 y_3 - x_3 y_2) \vec{e}_1 \\ &\quad + (x_3 y_1 - x_1 y_3) \vec{e}_2 \\ &\quad + (x_1 y_2 - x_2 y_1) \vec{e}_3 \end{aligned} \quad (1.7a)$$

$$\|\vec{x} \times \vec{y}\| = x y \sin \varphi \quad (1.7b)$$

The cross product of the two vector  $\vec{x}$  and  $\vec{y}$  results in a vector perpendicular to both  $\vec{x}$  and  $\vec{y}$ . This operation is only defined in 3-dimensional Cartesian space. The angle  $\varphi$  is the angle in space between  $\vec{x}$  and  $\vec{y}$ . The cross product can also be defined in terms of a matrix-vector operation  $\vec{x} \times \vec{y} \equiv \underline{\underline{\vec{x}}} \cdot \vec{y}$

$$\text{Cross product tensor:} \quad \underline{\underline{\vec{x}}} \equiv \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \quad (1.8)$$

Various identities for the cross product tensor can be verified. These identities will be extensively used throughout this dissertation.

$$\begin{aligned} \left[\underline{\underline{\vec{x}}}\right]^T &= -\underline{\underline{\vec{x}}} & \left[\underline{\underline{\vec{x}}}\right]^2 &= \vec{x} \cdot \underline{\underline{\vec{x}}} - x^2 \underline{\underline{\mathbf{I}}} & \underline{\underline{\underline{\vec{x} + \vec{u}}}} &= \underline{\underline{\vec{x}}} + \underline{\underline{\vec{u}}} \\ \underline{\underline{\vec{x}}} \cdot \underline{\underline{\vec{u}}} &= -\underline{\underline{\vec{u}}} \cdot \underline{\underline{\vec{x}}} & \left[\underline{\underline{\vec{x}}}\right]^3 &= -x^2 \underline{\underline{\vec{x}}} & \underline{\underline{\underline{\vec{x}} \cdot \vec{u}}} &= \underline{\underline{\vec{x}}} \cdot \underline{\underline{\vec{u}}} - \underline{\underline{\vec{u}}} \cdot \underline{\underline{\vec{x}}} \end{aligned} \quad (1.9)$$

with  $\underline{\underline{\mathbf{I}}}$  the  $3 \times 3$  identity matrix.

$$\text{Identity matrix:} \quad \underline{\underline{\mathbf{I}}} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.10)$$

## 1.2 Vector transformations

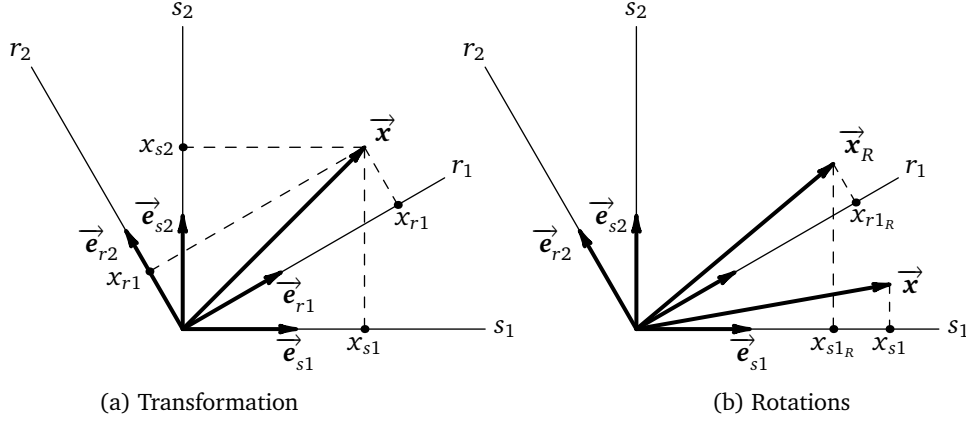
In this section only a basic overview of vector rotations and transformations is given to establish the basic nomenclature and definitions. For a more in-depth discussion refer to Hassenpflug (1993).

Consider two Cartesian axis systems denoted by  $s$  and  $r$  as shown in figure 1(a). From the general definition of a vector, equation (1.1), it follows

$$\vec{x} = [\vec{e}_{s1} \quad \vec{e}_{s2} \quad \vec{e}_{s3}] \cdot \begin{bmatrix} x_{s1} \\ x_{s2} \\ x_{s3} \end{bmatrix} = \underline{\underline{\mathbf{E}}}_s \cdot \vec{x}^s \quad (1.11)$$

The quantity,  $\underline{\underline{\mathbf{E}}}_s = [\vec{e}_{s1} \quad \vec{e}_{s2} \quad \vec{e}_{s3}]$ , is the base of the axis system denoted by  $s$ . It consists of the three orthogonal vectors parallel to the axes. From the outer product, equation (1.6), follows for the inverse of base  $\underline{\underline{\mathbf{E}}}_s$ :

$$\left[\underline{\underline{\mathbf{E}}}_s\right]^T \cdot \underline{\underline{\mathbf{E}}}_s = \underline{\underline{\mathbf{E}}}_s^s \cdot \underline{\underline{\mathbf{E}}}_s = \underline{\underline{\mathbf{I}}} \quad \Rightarrow \quad \left[\underline{\underline{\mathbf{E}}}_s\right]^T = \left[\underline{\underline{\mathbf{E}}}_s\right]^{-1} = \underline{\underline{\mathbf{E}}}_s^s \quad (1.12)$$



**Figure 1:** Vector transformations

We can repeat the procedure of equation (1.11) for the vector  $\vec{x}$  in terms of base  $\vec{E}_r$ . The relationship of the elements of vector  $\vec{x}$  in terms of base  $\vec{E}_s$  and base  $\vec{E}_r$  is then

$$\vec{x} = \vec{E}_r \cdot \vec{x}^r = \vec{E}_s \cdot \vec{x}^s \quad \Rightarrow \quad \begin{cases} \vec{x}^s = \vec{E}_s \cdot \vec{E}_r \cdot \vec{x}^r = \vec{E}_r^s \cdot \vec{x}^r \\ \vec{x}^r = \vec{E}_r \cdot \vec{E}_s \cdot \vec{x}^s = \vec{E}_s^r \cdot \vec{x}^s \end{cases} \quad (1.13)$$

The matrix quantities  $\vec{E}_r^s$  and  $\vec{E}_s^r$  are then the transformation matrices of the components of a vector between the two bases  $\vec{E}_s$  and  $\vec{E}_r$ . The columns of the transformation matrix  $\vec{E}_r^s$  are the elements of the unit vector  $\vec{e}_{s_i}$  expressed in base  $\vec{E}_r$  and the rows are the unit vectors  $\vec{e}_{r_j}$  expressed in base  $\vec{E}_s$ .

$$\vec{E}_r^s = [\vec{e}_{r1}^s \quad \vec{e}_{r2}^s \quad \vec{e}_{r3}^s] = \begin{bmatrix} \vec{e}_{r1}^{s1} \\ \vec{e}_{r1}^{s2} \\ \vec{e}_{r1}^{s3} \end{bmatrix} \quad (1.14)$$

The properties of the transformation matrix are well known, for example

$$[\vec{E}_r^s]^T = [\vec{E}_s^r]^{-1} = \vec{E}_s^r \quad (1.15)$$

The transformation matrix is often called *Direction Cosine matrix*

$$\vec{C}_s^r = \vec{E}_s^r = \vec{E}_s \cdot \vec{E}_r = [\vec{e}_{s_i} \cdot \vec{e}_{r_j}] = [\cos \varphi_{ij}] \quad (1.16)$$

### 1.3 Vector rotations

Consider the case of a vector in space with initial position  $\vec{x}$ . The vector is rotated to a new position in space,  $\vec{x}_R$ . Define the rotation tensor<sup>1</sup> operation then as

$$\vec{x}_R = \vec{R} \cdot \vec{x} \quad (1.17)$$

<sup>1</sup>A large body of literature define the rotation tensor as  $\vec{x} = \vec{R} \cdot \vec{x}_R$ , where  $[\vec{R}]^T = \vec{R}$ .

If the operation is applied to the rotation of all the direction vectors of a base  $\vec{\underline{E}}_s$  to a new rotated base  $\vec{\underline{E}}_r$ , then

$$\vec{\underline{E}}_r = \vec{\underline{R}} \cdot \vec{\underline{E}}_s \quad (1.18)$$

or

$$\vec{\underline{E}}_r^s = \vec{\underline{E}}_s^s \cdot \vec{\underline{R}} \cdot \vec{\underline{E}}_s = \vec{\underline{R}}^s \quad (1.19)$$

With reference to figure 1(b), consider the case of a vector fixed in a rotating base  $\vec{\underline{E}}_r$  with initial position  $\vec{x}$  and final position after a rotation of  $\vec{x}_R$ . If the initial orientation of  $\vec{\underline{E}}_r$  corresponds with that of  $\vec{\underline{E}}_s$  then the numerical values of the components of  $\vec{x}^s$  and  $\vec{x}_R^r$  are equal. From the transformation of  $\vec{x}_R^r$  it then follows that

$$\vec{x}_R^s = \vec{\underline{E}}_r^s \cdot \vec{x}_R^r = \vec{\underline{R}}^s \cdot \vec{x}^s \quad (1.20)$$

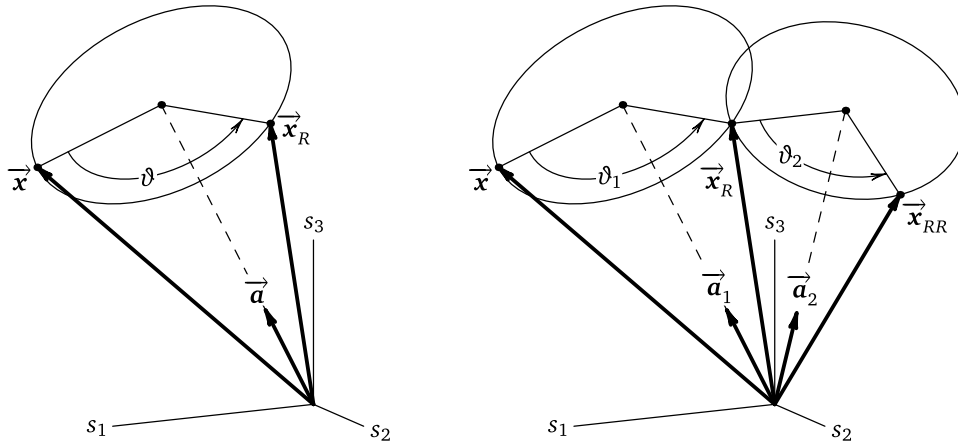
If the rotation matrix is transformed between bases, then

$$\vec{\underline{R}}_r^r = \vec{\underline{E}}_s^r \cdot \vec{\underline{R}}^s \cdot \vec{\underline{E}}_r^s = \vec{\underline{R}}^s \quad (1.21)$$

The rotation matrix is therefore identical in terms of both bases and we can thus denote it without the base indices, except when there is more than one rotation. The rotation matrix between bases  $\vec{\underline{E}}_s$  and  $\vec{\underline{E}}_r$  in terms of the transformation matrix is given by

$$\vec{\underline{R}} = \vec{\underline{E}}_r^s \quad (1.22)$$

$$[\vec{\underline{R}}]^{-1} = [\vec{\underline{R}}]^T = \vec{\underline{E}}_s^r \quad (1.23)$$



(a) Single rotation

(b) Multiple rotations

**Figure 2:** General vector rotation

## 2 General rotations

### 2.1 The general rotation matrix (Rodriguez formula)

Euler's theorem states that the most general displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis through that point. With reference to figure 2, consider a vector with initial position  $\vec{x}$ . The vector is rotated about an axis defined by the unit vector  $\vec{a}$ , through an angle  $\vartheta$ . The vector after rotation is denoted by  $\vec{x}_R$ . From the geometry in figure 2(a) it can be shown (e.g., Shabana 1998, §2.1) for the vector components in terms of the stationary base  $\underline{\underline{E}}_s$  that

$$\vec{x}_R^s = \vec{x}^s + \sin \vartheta (\vec{a}^s \times \vec{x}^s) + (1 - \cos \vartheta) (\vec{a}^s \times (\vec{a}^s \times \vec{x}^s)) \quad (2.1)$$

Rewrite equation (2.1) in terms of the cross product tensor defined in equation (1.8)

$$\vec{x}_R^s = \left[ \underline{\underline{I}} + \sin \vartheta \underline{\underline{a}}_s^s + (1 - \cos \vartheta) \underline{\underline{a}}_s^s \cdot \underline{\underline{a}}_s^s \right] \cdot \vec{x}^s \quad (2.2)$$

with  $\underline{\underline{I}}$  is the  $3 \times 3$  unit matrix. By comparing equations (2.2) and (1.21), the general format of the rotation matrix for a rotation through an angle  $\vartheta$  about an axis  $\vec{a}^s$  fixed in base  $\underline{\underline{E}}_s$  is given by

$$\underline{\underline{R}} = \underline{\underline{I}} + \sin \vartheta \underline{\underline{a}}_s^s + (1 - \cos \vartheta) \underline{\underline{a}}_s^s \cdot \underline{\underline{a}}_s^s \quad (2.3)$$

$$[\underline{\underline{R}}]^T = \underline{\underline{I}} - \sin \vartheta \underline{\underline{a}}_s^s + (1 - \cos \vartheta) \underline{\underline{a}}_s^s \cdot \underline{\underline{a}}_s^s \quad (2.4)$$

Equation (2.3) is also known as the *Rodriguez formula*. Note that four scalar parameters ( $\vartheta$  and the three components of  $\vec{a}$ ) and the constraint  $\|\vec{a}\|=1$  describe three degrees of rotational freedom.

If  $\vec{x}$  is fixed to a rotating base  $\underline{\underline{E}}_r$ , with  $\vec{x}^s = \vec{x}_R^r$  (see figure 1(b)), then  $\underline{\underline{E}}_r^s$  is the transformation matrix from base  $\underline{\underline{E}}_r$  to base  $\underline{\underline{E}}_s$  and

$$\underline{\underline{E}}_r^s = \underline{\underline{R}} \quad \text{and} \quad \underline{\underline{E}}_s^r = [\underline{\underline{R}}]^T \quad (2.5)$$

Note for the transformation of the cross product tensor associated with the rotation axis, is  $\underline{\underline{a}}_s^s = \underline{\underline{a}}_r^r = \underline{\underline{a}}$ , because the components are identical in both the bases.

Equation (2.3) can also be written in exponential format by expanding  $\sin \vartheta$  and  $\cos \vartheta$  as Taylor series

$$\begin{aligned} \sin \vartheta &= \vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} + \dots \\ \cos \vartheta &= 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} + \dots \end{aligned} \quad (2.6)$$

With the aid of equation (1.9) follows the elegant solution by Argyris (1982)

$$\begin{aligned} \underline{\underline{R}} &= \underline{\underline{I}} + \left( \vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} + \dots \right) \underline{\underline{a}} + \left( \frac{\vartheta^2}{2!} - \frac{\vartheta^4}{4!} + \dots \right) [\underline{\underline{a}}]^2 \\ &= \underline{\underline{I}} + \vartheta \underline{\underline{a}} + \frac{\vartheta^2}{2!} [\underline{\underline{a}}]^2 + \frac{\vartheta^3}{3!} [\underline{\underline{a}}]^3 + \dots + \frac{\vartheta^n}{n!} [\underline{\underline{a}}]^n + \dots \end{aligned} \quad (2.7)$$

which is the exponential matrix

$$\underline{\bar{\mathbf{R}}} = \mathbf{e}^{\vartheta \underline{\bar{\mathbf{a}}}} \quad \text{and} \quad [\underline{\bar{\mathbf{R}}}]^T = \mathbf{e}^{-\vartheta \underline{\bar{\mathbf{a}}}} \quad (2.8)$$

For numerical purposes equation (2.3) can be written as a single matrix. Let  $c = \cos \vartheta$  and  $s = \sin \vartheta$ , then the rotation or transformation matrix is given by

$$\underline{\bar{\mathbf{R}}} = \underline{\bar{\mathbf{E}}}_r^s = \begin{bmatrix} a_1^2(1-c)+c & a_1 a_2(1-c)-a_3 s & a_1 a_3(1-c)+a_2 s \\ a_1 a_2(1-c)+a_3 s & a_2^2(1-c)+c & a_2 a_3(1-c)-a_1 s \\ a_1 a_3(1-c)-a_2 s & a_2 a_3(1-c)+a_1 s & a_3^2(1-c)+c \end{bmatrix} \quad (2.9)$$

Useful rotation matrices are the three rotations  $\vartheta_1$ ,  $\vartheta_2$  and  $\vartheta_3$  about the unit vectors in the axis directions  $\underline{\bar{\mathbf{e}}}_1=[1 \ 0 \ 0]^T$ ,  $\underline{\bar{\mathbf{e}}}_2=[0 \ 1 \ 0]^T$  and  $\underline{\bar{\mathbf{e}}}_3=[0 \ 0 \ 1]^T$ .

$$\underline{\bar{\mathbf{R}}}_1 = \underline{\bar{\mathbf{R}}}(\vartheta_1, \underline{\bar{\mathbf{e}}}_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta_1 & -\sin \vartheta_1 \\ 0 & \sin \vartheta_1 & \cos \vartheta_1 \end{bmatrix} \quad (2.10a)$$

$$\underline{\bar{\mathbf{R}}}_2 = \underline{\bar{\mathbf{R}}}(\vartheta_2, \underline{\bar{\mathbf{e}}}_2) = \begin{bmatrix} \cos \vartheta_2 & 0 & \sin \vartheta_2 \\ 0 & 1 & 0 \\ -\sin \vartheta_2 & 0 & \cos \vartheta_2 \end{bmatrix} \quad (2.10b)$$

$$\underline{\bar{\mathbf{R}}}_3 = \underline{\bar{\mathbf{R}}}(\vartheta_3, \underline{\bar{\mathbf{e}}}_3) = \begin{bmatrix} \cos \vartheta_3 & -\sin \vartheta_3 & 0 \\ \sin \vartheta_3 & \cos \vartheta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.10c)$$

It is frequently necessary to find the rotation axis  $\underline{\bar{\mathbf{a}}}$  and rotation angle  $\vartheta$  for a known transformation matrix,  $\underline{\bar{\mathbf{E}}}_r^s = [E_{ij}]$ . From equation (2.9) various relationships can be deduced. Two of the more important ones are

$$2 \cos \vartheta = E_{11} + E_{22} + E_{33} - 1 \quad (2.11)$$

$$2 \sin \vartheta \underline{\bar{\mathbf{a}}} = \begin{bmatrix} E_{32} - E_{23} \\ E_{13} - E_{31} \\ E_{21} - E_{12} \end{bmatrix} \quad (2.12)$$

When  $\vartheta \approx \pi$  equation (2.12) can not be used to find  $\underline{\bar{\mathbf{a}}}$ . Another more general approach, is to consider the characteristic polynomial of  $\underline{\bar{\mathbf{E}}}_r^s$ .

$$\det[\underline{\bar{\mathbf{E}}}_r^s - \lambda \underline{\bar{\mathbf{I}}}] = (\lambda^2 + 2\lambda \cos \vartheta + 1)(1 - \lambda) = 0 \quad (2.13)$$

It leads to the eigenvalues  $\lambda = e^{i\vartheta}$ ,  $e^{-i\vartheta}$ , 1. It can therefore be stated that  $\lambda = 1$  is always an eigenvalue of  $\underline{\bar{\mathbf{E}}}_r^s$  and that an eigenvector or axis  $\underline{\bar{\mathbf{a}}} = \underline{\bar{\mathbf{a}}}^s = \underline{\bar{\mathbf{a}}}^r$  exists that is unchanged by the rotation. The rotation axis can be obtained with a numerical method by solving the eigenvector problem  $\underline{\bar{\mathbf{E}}}_r^s \cdot \underline{\bar{\mathbf{a}}} = \underline{\bar{\mathbf{a}}}$ .

## 2.2 Multiple rotations

For the case of multiple rotations of a vector as shown in figure 2(b), let

$$\underline{\bar{x}}_R^s = \underline{\bar{R}}_1 \cdot \underline{\bar{x}}^s \quad \text{with} \quad \underline{\bar{R}}_1 = \underline{\bar{R}}(\vartheta_1, \underline{\bar{a}}_1) \quad (2.14)$$

$$\underline{\bar{x}}_{RR}^s = \underline{\bar{R}}_2 \cdot \underline{\bar{x}}_R^s \quad \text{with} \quad \underline{\bar{R}}_2 = \underline{\bar{R}}(\vartheta_2, \underline{\bar{a}}_2) \quad (2.15)$$

then

$$\underline{\bar{x}}_{RR}^s = \underline{\bar{R}}_2 \cdot \underline{\bar{x}}_R^s = \underline{\bar{R}}_2 \cdot \underline{\bar{R}}_1 \cdot \underline{\bar{x}}^s = \underline{\bar{R}} \cdot \underline{\bar{x}}^s \quad (2.16)$$

with

$$\underline{\bar{R}} = \underline{\bar{R}}_2 \cdot \underline{\bar{R}}_1 \quad (2.17)$$

If  $\underline{\bar{x}}$  is fixed to a rotating base  $\underline{\bar{E}}_r$ , with  $\underline{\bar{x}}^s = \underline{\bar{x}}_{RR}^r$ , then  $\underline{\bar{E}}_r^s$  is the transformation matrix from base  $\underline{\bar{E}}_r$  to base  $\underline{\bar{E}}_s$  and

$$\underline{\bar{E}}_r^s = \underline{\bar{R}}_2 \cdot \underline{\bar{R}}_1 \quad (2.18)$$

$$\underline{\bar{E}}_s^r = \left[ \underline{\bar{R}}_2 \cdot \underline{\bar{R}}_1 \right]^T = \left[ \underline{\bar{R}}_1 \right]^T \cdot \left[ \underline{\bar{R}}_2 \right]^T \quad (2.19)$$

Note that in general is  $\underline{\bar{R}}_1 \cdot \underline{\bar{R}}_2 \neq \underline{\bar{R}}_2 \cdot \underline{\bar{R}}_1$ . If we write equation (2.19) in terms of the exponential representation of equation (2.8) then

$$\underline{\bar{R}} = e^{\vartheta_1 \underline{\bar{a}}_1} \cdot e^{\vartheta_2 \underline{\bar{a}}_2} \neq e^{(\vartheta_1 \underline{\bar{a}}_1 + \vartheta_2 \underline{\bar{a}}_2)} \quad (2.20)$$

This means that the rotations are not vectors that can be added. The only exception is when the rotation axes are parallel,  $\underline{\bar{a}}_1 \parallel \underline{\bar{a}}_2$ .

## 2.3 Infinitesimal rotations

In case of an infinitesimal rotation  $\Delta\vartheta$ , second and higher order terms in the series expansion in equation (2.7) can be neglected, resulting in

$$\Delta \underline{\bar{R}} \approx \underline{\bar{I}} + \Delta\vartheta \underline{\bar{a}} \quad \text{and} \quad \left[ \Delta \underline{\bar{R}} \right]^T \approx \underline{\bar{I}} - \Delta\vartheta \underline{\bar{a}} \quad (2.21)$$

In the previous section it was proven that finite rotations are not vector quantities that can be added. Infinitesimal rotation are vector quantities that can be added to give a total rotation. Consider two infinitesimal rotations  $\Delta\vartheta_1$  and  $\Delta\vartheta_2$  about axes  $\underline{\bar{a}}_1$  and  $\underline{\bar{a}}_2$

$$\Delta \underline{\bar{R}}_1 = \underline{\bar{I}} + \Delta\vartheta_1 \underline{\bar{a}}_1 \quad \text{and} \quad \Delta \underline{\bar{R}}_2 = \underline{\bar{I}} + \Delta\vartheta_2 \underline{\bar{a}}_2 \quad (2.22)$$

For a multiple rotation

$$\begin{aligned} \Delta \underline{\bar{R}}_1 \cdot \Delta \underline{\bar{R}}_2 &= \left[ \underline{\bar{I}} + \Delta\vartheta_1 \underline{\bar{a}}_1 \right] \cdot \left[ \underline{\bar{I}} + \Delta\vartheta_2 \underline{\bar{a}}_2 \right] \\ &= \underline{\bar{I}} + \Delta\vartheta_1 \underline{\bar{a}}_1 + \Delta\vartheta_2 \underline{\bar{a}}_2 + \Delta\vartheta_1 \Delta\vartheta_2 \underline{\bar{a}}_1 \cdot \underline{\bar{a}}_2 \\ &\approx \underline{\bar{I}} + \Delta\vartheta_1 \underline{\bar{a}}_1 + \Delta\vartheta_2 \underline{\bar{a}}_2 \end{aligned} \quad (2.23)$$



where second and higher order terms were again ignored. This results in

$$\underline{\Delta \mathbf{R}}_1 \cdot \underline{\Delta \mathbf{R}}_2 = \underline{\Delta \mathbf{R}}_2 \cdot \underline{\Delta \mathbf{R}}_1 \quad (2.24)$$

proving that two successive infinitesimal rotations about different axes can be added and that an infinitesimal rotation is a vector. For  $n$  successive rotations it can be shown that

$$\begin{aligned} \underline{\Delta \mathbf{R}}_1 \cdot \underline{\Delta \mathbf{R}}_2 \cdots \underline{\Delta \mathbf{R}}_n &= \prod_{i=1}^n \underline{\Delta \mathbf{R}}_i = \underline{\mathbf{I}} + \Delta\vartheta_1 \underline{\mathbf{a}}_1 + \Delta\vartheta_2 \underline{\mathbf{a}}_2 + \cdots + \Delta\vartheta_n \underline{\mathbf{a}}_n \\ &= \underline{\mathbf{I}} + \sum_{i=1}^n \Delta\vartheta_i \underline{\mathbf{a}}_i \\ &= \underline{\Delta \mathbf{R}}_n \cdot \underline{\Delta \mathbf{R}}_{n-1} \cdots \underline{\Delta \mathbf{R}}_1 \end{aligned} \quad (2.25)$$

## 2.4 Angular velocity

Consider three successive infinitesimal rotations  $\Delta\vartheta_1$ ,  $\Delta\vartheta_2$  and  $\Delta\vartheta_3$  about the unit vectors in the axis directions  $\underline{\mathbf{e}}_1^s = [1 \ 0 \ 0]^T$ ,  $\underline{\mathbf{e}}_2^s = [0 \ 1 \ 0]^T$  and  $\underline{\mathbf{e}}_3^s = [0 \ 0 \ 1]^T$ . The total infinitesimal rotation is then from equation (2.25)

$$\underline{\Delta \mathbf{R}} = \underline{\Delta \mathbf{R}}_1 \cdot \underline{\Delta \mathbf{R}}_2 \cdot \underline{\Delta \mathbf{R}}_3 = \underline{\mathbf{I}} + \underline{\Delta \vartheta}^s \quad (2.26)$$

with

$$\underline{\Delta \vartheta}^s = \Delta\vartheta_1 \underline{\mathbf{e}}_1^s + \Delta\vartheta_2 \underline{\mathbf{e}}_2^s + \Delta\vartheta_3 \underline{\mathbf{e}}_3^s = \begin{bmatrix} 0 & -\Delta\vartheta_3 & \Delta\vartheta_2 \\ \Delta\vartheta_3 & 0 & -\Delta\vartheta_1 \\ -\Delta\vartheta_2 & \Delta\vartheta_1 & 0 \end{bmatrix} \quad (2.27)$$

The total infinitesimal rotation of a fixed length vector  $\underline{\xi}^s$  about three mutually perpendicular axes is then

$$\underline{\xi}_\Delta^s = \underline{\Delta \mathbf{R}} \cdot \underline{\xi}^s = \left[ \underline{\mathbf{I}} + \underline{\Delta \vartheta}^s \right] \cdot \underline{\xi}^s = \underline{\xi}^s + \underline{\Delta \vartheta}^s \cdot \underline{\xi}^s \quad (2.28)$$

The change vector is

$$\underline{\Delta \xi}^s = \underline{\xi}_\Delta^s - \underline{\xi}^s = \underline{\Delta \vartheta}^s \cdot \underline{\xi}^s \quad (2.29)$$

Divide equation (2.29) by the time increment  $\Delta t$  during which the rotations take place. For the limit as  $\Delta t$  approaches zero follows

$$\lim_{\Delta t \rightarrow 0} \frac{\underline{\Delta \xi}^s}{\Delta t} = \frac{d\underline{\xi}^s}{dt} = \dot{\underline{\xi}}^s \quad (2.30)$$

and

$$\lim_{\Delta t \rightarrow 0} \frac{\underline{\Delta \vartheta}^s}{\Delta t} \cdot \underline{\xi}^s = \underline{\omega}^s \cdot \underline{\xi}^s = \underline{\omega}^s \times \underline{\xi}^s \quad (2.31)$$

so that the time derivative of a rotating vector of fixed length become

$$\dot{\bar{\xi}}^s = \bar{\omega}^s \cdot \bar{\xi}^s = \bar{\omega}^s \times \bar{\xi}^s \quad (2.32)$$

The vector  $\bar{\omega}$  is defined as the angular velocity with components the instantaneous rotation rate about the three coordinate axes

$$\omega_i = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vartheta_i}{\Delta t} \quad i = 1, 2, 3 \quad (2.33)$$

It must emphasized that the angular velocity is a defined vector and *not the derivative of any quantity*. This implies that the angular velocity cannot be integrated to the obtain the attitude or orientations of a vector or base or any other quantity.

As an application of equation (2.32), the time derivative of a transformation matrix in equation (1.14) from a static base  $\bar{\underline{E}}_s$  to a rotating base  $\bar{\underline{E}}_r$  can be obtained

$$\dot{\bar{\underline{E}}}_r^s = [\dot{\bar{e}}_{r1}^s \quad \dot{\bar{e}}_{r2}^s \quad \dot{\bar{e}}_{r3}^s] = \left[ \bar{\omega}^s \cdot \bar{e}_{r1}^s \quad \bar{\omega}^s \cdot \bar{e}_{r2}^s \quad \bar{\omega}^s \cdot \bar{e}_{r3}^s \right] = \bar{\omega}^s \cdot \bar{\underline{E}}_r^s \quad (2.34)$$

or for the angular velocity in terms of the rotating base  $\bar{\underline{E}}_r$

$$\dot{\bar{\underline{E}}}_r^s = \left[ \bar{\underline{E}}_r^s \cdot \bar{\omega}^r \cdot \bar{\underline{E}}_r^r \right] \cdot \bar{\underline{E}}_r^s = \bar{\underline{E}}_r^s \cdot \bar{\omega}^r \quad (2.35)$$

Define the vectors  $\bar{x}^s$  and  $\dot{\bar{x}}^s = d\bar{x}^s/dt$  as the position and velocity of a particle or point with components in terms of a static base  $\bar{\underline{E}}_s$ , while  $\bar{x}^r$  and  $\dot{\bar{x}}^r$  are the position and apparent velocity and acceleration in terms of a rotating base  $\bar{\underline{E}}_r$ .

$$\bar{x}^s = \bar{\underline{E}}_r^s \cdot \bar{x}^r \quad (2.36)$$

$$\dot{\bar{x}}^s = \bar{\underline{E}}_r^s \cdot \dot{\bar{x}}^r + \dot{\bar{\underline{E}}}_r^s \cdot \bar{x}^r = \bar{\underline{E}}_r^s \cdot \left[ \dot{\bar{x}}^r + \bar{\underline{E}}_r^r \cdot \dot{\bar{\underline{E}}}_r^s \cdot \bar{x}^r \right] = \bar{\underline{E}}_r^s \cdot \left[ \dot{\bar{x}}^r + \bar{\omega}^r \times \bar{x}^r \right] \quad (2.37)$$

$$\ddot{\bar{x}}^s = \bar{\underline{E}}_r^s \cdot \left[ \ddot{\bar{x}}^r + 2\bar{\omega}^r \times \dot{\bar{x}}^r + \dot{\bar{\omega}}^r \times \bar{x}^r + \bar{\omega}^r \times (\bar{\omega}^r \times \bar{x}^r) \right] \quad (2.38)$$

Note that  $\dot{\bar{\omega}}^s = \bar{\underline{E}}_r^s \cdot \dot{\bar{\omega}}^r$  because  $\bar{\omega}^r \times \bar{\omega}^r = \bar{0}$  The cross product tensor of the angular velocity  $\bar{\omega}$  is from equation (2.34)

$$\begin{aligned} \bar{\omega}^r &= \bar{\underline{E}}_r^r \cdot \dot{\bar{\underline{E}}}_r^s \\ \bar{\omega}^s &= \bar{\underline{E}}_r^s \cdot \bar{\omega}^r \cdot \bar{\underline{E}}_r^r = \dot{\bar{\underline{E}}}_r^s \cdot \bar{\underline{E}}_r^r \end{aligned} \quad (2.39)$$

## 2.5 Rotation kinematics

We proceed next to obtain  $\vec{\omega}$  as a function of  $\vec{a}$  and  $\vartheta$ . The following identities can then be verified from the fact that  $\vec{a}$  is a unit vector, ( $\underline{a} \cdot \bar{a} = 1$ ), implying that ( $\underline{a} \cdot \dot{\bar{a}} = 0$ ):

$$\begin{aligned}\bar{\underline{a}} \cdot \dot{\bar{\underline{a}}} \cdot \bar{\underline{a}} &= -(\underline{a} \cdot \dot{\bar{a}}) \bar{\underline{a}} = \underline{0} \\ \bar{\underline{a}} \cdot \bar{\underline{a}} \cdot \dot{\bar{\underline{a}}} \cdot \bar{\underline{a}} &= -(\underline{a} \cdot \dot{\bar{a}}) \bar{\underline{a}} \cdot \bar{\underline{a}} = \underline{0}\end{aligned}\quad (2.40)$$

The angular velocity tensor in equation (2.39), after the differentiation of the transformation matrix equation (2.3) and algebraic manipulation with the aid of equations (2.40) and (1.9) is

$$\begin{aligned}\bar{\underline{\omega}}^r &= \vartheta \dot{\bar{\underline{a}}} + \sin \vartheta \dot{\bar{\underline{a}}} - 2 \sin^2 \frac{\vartheta}{2} \left[ \bar{\underline{a}} \cdot \dot{\bar{\underline{a}}} - \dot{\bar{\underline{a}}} \cdot \bar{\underline{a}} \right] \\ &= \vartheta \dot{\bar{\underline{a}}} + \sin \vartheta \dot{\bar{\underline{a}}} - 2 \sin^2 \frac{\vartheta}{2} \overline{\underline{a} \cdot \dot{\bar{a}}}\end{aligned}\quad (2.41)$$

From equation (2.41), the vector equation for  $\bar{\omega}^r$  and  $\bar{\omega}^s$  (where the latter can be derived with the same arguments), follows then as

$$\begin{aligned}\bar{\omega}^r &= \vartheta \dot{\bar{a}} + \sin \vartheta \dot{\bar{a}} - 2 \sin^2 \frac{\vartheta}{2} \bar{\underline{a}} \cdot \dot{\bar{\underline{a}}} \\ \bar{\omega}^s &= \vartheta \dot{\bar{a}} + \sin \vartheta \dot{\bar{a}} + 2 \sin^2 \frac{\vartheta}{2} \bar{\underline{a}} \cdot \dot{\bar{\underline{a}}}\end{aligned}\quad (2.42)$$

The inner or scalar product of equation (2.42) gives the norm of the angular velocity

$$\omega^2 = \underline{\omega}_r \cdot \bar{\omega}^r = \underline{\omega}_s \cdot \bar{\omega}^s = \vartheta^2 + 4 \sin^2 \frac{\vartheta}{2} \dot{a}^2 \quad (2.43)$$

From equation (2.42) the time derivative of the rotation angle  $\vartheta$  is

$$\dot{\vartheta} = \underline{a} \cdot \bar{\omega}^r = \underline{a} \cdot \bar{\omega}^s \quad (2.44)$$

Multiply equation (2.44) with  $\bar{a}$ . With the aid of the triple cross-product identities, it then follows

$$\begin{aligned}\dot{\vartheta} \bar{a} &= (\underline{a} \cdot \bar{\omega}^r) \bar{a} = \bar{\omega}^r + \bar{\underline{a}} \cdot \bar{\underline{a}} \cdot \bar{\omega}^r \\ &= (\underline{a} \cdot \bar{\omega}^s) \bar{a} = \bar{\omega}^s + \bar{\underline{a}} \cdot \bar{\underline{a}} \cdot \bar{\omega}^s\end{aligned}\quad (2.45)$$

Inspection of equations (2.42) to (2.44) reveals that  $\underline{a} \cdot \vec{\omega} = \dot{\vartheta} \neq \omega$ . The angular velocity vector  $\vec{\omega}$  is therefore in general not in the direction of the instantaneous rotation axis  $\vec{a}$ .

The vector  $\dot{\bar{a}}$  can be obtained from equation (2.42) by the substitution of equation (2.45) and assuming a solution of the form  $[\bar{\underline{I}} + \alpha \bar{\underline{a}} + \beta \bar{\underline{a}} \cdot \bar{\underline{a}}]$ . With the aid of the identities in equations (2.40) and (1.9), it leads to

$$\begin{aligned}\dot{\bar{a}} &= \frac{1}{2} \left[ +\bar{\underline{a}} - \cot \frac{\vartheta}{2} \bar{\underline{a}} \cdot \bar{\underline{a}} \right] \cdot \bar{\omega}^r \equiv \bar{\underline{K}}_r \cdot \bar{\omega}^r \\ &= \frac{1}{2} \left[ -\bar{\underline{a}} - \cot \frac{\vartheta}{2} \bar{\underline{a}} \cdot \bar{\underline{a}} \right] \cdot \bar{\omega}^s \equiv \bar{\underline{K}}_s \cdot \bar{\omega}^s\end{aligned}\quad (2.46)$$

Note the notation in equation (2.46) for  $\underline{\bar{K}}_r$ . It is a tensor in a mixed base (see Hassenpflug 1993), because  $\bar{\mathbf{a}}^r = \bar{\mathbf{a}}^s$ . For the transformation between bases it can also be confirmed that

$$\underline{\bar{K}}_r = \underline{\bar{K}}_s \cdot \underline{\bar{E}}_r^s \quad (2.47)$$

The general kinematic equations for a rotating base are given by equations (2.44) and (2.46). The four scalar equations describe only three degrees of freedom and are constrained by  $\|\bar{\mathbf{a}}\| = 1$ . These equations can be integrated to obtain  $\underline{\bar{E}}_r^s$  as a functions of time, but equation (2.46) is singular for values of  $\vartheta = 0, \pm 2\pi, \dots$ , which render a general numeric solution impractical.

### 3 Attitude determination

#### 3.1 Direct integration

The classic problem in rotation kinematics is that the angular velocity cannot be integrated to obtain the orientation of a rotating base, because the integral is dependent on the path of integration. The most basic method to find the orientation of  $\vec{E}_r$  as a function of time is to integrate equation (2.35) directly,

$$\begin{aligned} \dot{\vec{E}}_r^s &= \vec{\omega}_s^s \cdot \vec{E}_r^s = \begin{bmatrix} \vec{\omega}_s^s \times \vec{e}_{r1}^s & \vec{\omega}_s^s \times \vec{e}_{r2}^s & \vec{\omega}_s^s \times \vec{e}_{r3}^s \end{bmatrix} \\ \dot{\vec{E}}_s^r &= -\vec{\omega}_r^r \cdot \vec{E}_s^r = -\begin{bmatrix} \vec{\omega}_r^r \times \vec{e}_{s1}^r & \vec{\omega}_r^r \times \vec{e}_{s2}^r & \vec{\omega}_r^r \times \vec{e}_{s3}^r \end{bmatrix} \end{aligned} \quad (3.1)$$

Only two of the vectors need to be integrated. The third vector can be obtained from the cross product ( $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$ ). This method involves six parameters while there are only three degrees of freedom. With a lot of effort and by careful selection of elements from the orthogonality constraint requirement  $\vec{E}_r^s \cdot \vec{E}_s^r = \vec{I}$ , it can be refined to three parameters. It is also advisable that the constraint equation be enforced through frequent normalization, to compensate for the fact that the constraints are not taken into account during integration.

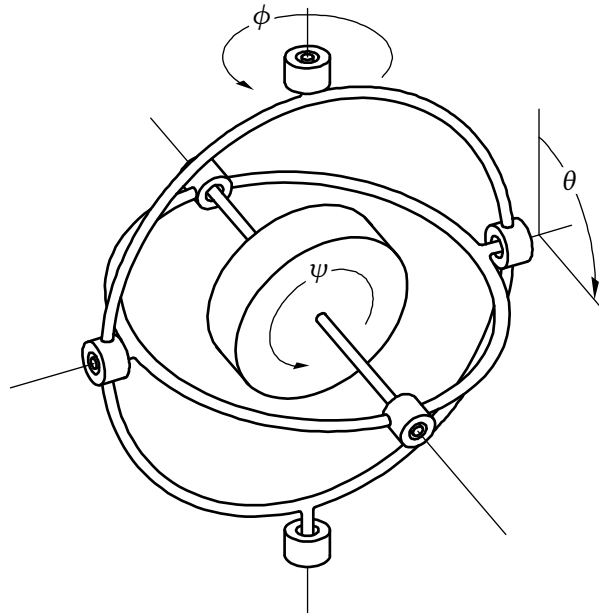


Figure 3: 3-1-3 Euler angle gyroscope representation

#### 3.2 Euler Angles

Three independent coordinates are needed to describe the orientation or attitude of a rotating rigid body in space. A class of parametrizations that enjoys wide acceptance is the *Euler Angles*. It consists of three successive rotations, see for example figure 3. There is no general rule for the sequence of the rotations about the different axes. In fact there exists twelve distinct combinations of rotations to transform a

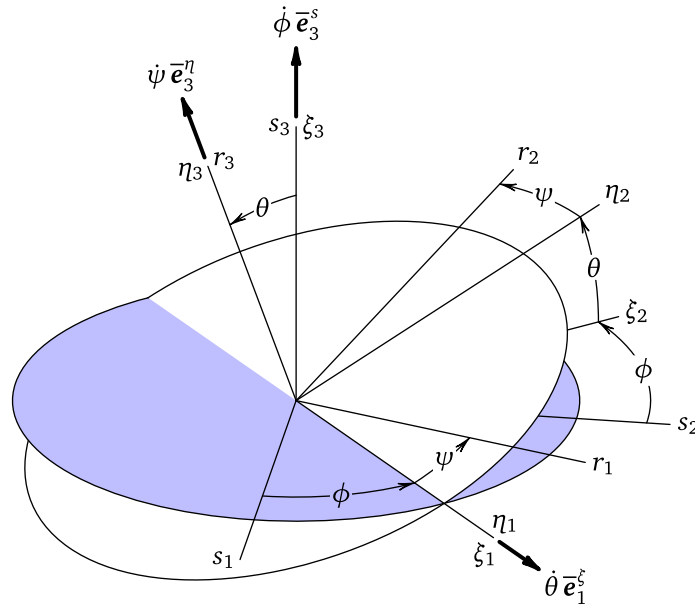
vector from one Cartesian axis system into another (e.g. Wertz 1978): six symmetric sets, whose labels are written as

$$1-2-1 \quad 1-3-1 \quad 2-1-2 \quad 2-3-2 \quad \boxed{3-1-3} \quad 3-2-3$$

and six asymmetric sets, designated by

$$1-2-3 \quad 1-3-2 \quad 2-1-3 \quad 2-3-1 \quad 3-1-2 \quad 3-2-1$$

The Euler angle formulations have kinematic singularities, but their geometrical significance is apparent. They are especially useful for analytical dynamics formulations to find analytical and perturbation solutions. Because of their geometrical significance they are often employed to define initial values for the rotation matrices.



**Figure 4:** 3-1-3 Euler angle scheme

### 3-1-3 Euler Angle Parametrization

Figure 4 is the representation of the 3-1-3 Euler angle scheme and it consists of three successive rotations:

- (1) A rotation  $\phi$  about the  $\bar{e}_3^s$  of base  $\bar{\underline{E}}_s$  to an intermediate base  $\bar{\underline{E}}_\xi$

$$\bar{\underline{E}}_\xi^s(\phi, \bar{e}_3^s) = \bar{\underline{R}}_s^s(\phi, \bar{e}_3^s) = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.2a)$$

- (2) A rotation  $\theta$  about the  $\bar{e}_1^\xi$  of base  $\bar{\underline{E}}_\xi$  to an intermediate base  $\bar{\underline{E}}_\eta$

$$\bar{\underline{E}}_\eta^\xi(\theta, \bar{e}_1^\xi) = \bar{\underline{R}}_\xi^\xi(\theta, \bar{e}_1^\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad (3.2b)$$

(3) A rotation  $\psi$  about the  $\bar{\mathbf{e}}_3^n$  of base  $\bar{\mathbf{E}}_\eta$  to the final base  $\bar{\mathbf{E}}_r$

$$\bar{\mathbf{E}}_r^n(\psi, \bar{\mathbf{e}}_3^n) = \bar{\mathbf{R}}_\eta^n(\psi, \bar{\mathbf{e}}_3^n) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.2c)$$

The three rotations, equations (3.2a) to (3.2c), can be combined to obtain the multiple rotation matrix in base  $\bar{\mathbf{E}}_s$  or the transformation matrix from base  $\bar{\mathbf{E}}_r$  to base  $\bar{\mathbf{E}}_s$

$$\overline{\mathbf{E}}_{313_r}^s(\phi, \theta, \psi) = \bar{\mathbf{E}}_\xi^s(\phi, \bar{\mathbf{e}}_3^s) \cdot \bar{\mathbf{E}}_\eta^\xi(\theta, \bar{\mathbf{e}}_1^\xi) \cdot \bar{\mathbf{E}}_r^\eta(\psi, \bar{\mathbf{e}}_3^n) \quad (3.3a)$$

$$\overline{\mathbf{R}}_{313_s}^s(\phi, \theta, \psi) = \bar{\mathbf{R}}_s^s(\phi, \bar{\mathbf{e}}_3^s) \cdot \bar{\mathbf{R}}_\xi^\xi(\theta, \bar{\mathbf{e}}_1^\xi) \cdot \bar{\mathbf{R}}_\eta^\eta(\psi, \bar{\mathbf{e}}_3^n) \quad (3.3b)$$

For the numerical purposes the transformation matrix, equation (3.3a) can be written as a single matrix. Let  $s = \sin$  and  $c = \cos$  then

$$\begin{aligned} \overline{\mathbf{E}}_{313_r}^s(\phi, \theta, \psi) &= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \cdot \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix} \end{aligned} \quad (3.4)$$

### 3-1-3 Euler angle initial values

It is often necessary to find the Euler angles from a known or given transformation matrix,  $\overline{\mathbf{E}}_{313_r}^s(\phi, \theta, \psi) = [E_{ij}]$ . The Euler angles for the 3-1-3 Euler Angle Scheme follows then from equation (3.4) as

$$\begin{aligned} \cos \theta &= E_{33} \\ \sin \theta &= \frac{E_{31}}{\sin \psi} = \frac{E_{32}}{\cos \psi} = \frac{E_{13}}{\sin \phi} = \frac{-E_{23}}{\cos \psi} \\ \tan \phi &= \frac{-E_{13}}{E_{23}} \\ \tan \psi &= \frac{E_{31}}{E_{32}} \end{aligned} \quad (3.5)$$

For  $\theta = 0, \pm\pi$  the angles  $\phi$  and  $\psi$  are not defined. It is also clear from the gyroscope representation in figure 3 that the outer and inner gimbals are then in the same plane, resulting in the locking of the gimbals, with a loss of a degree of freedom. The combined angles  $\phi + \psi$  and  $\phi - \psi$  are always defined and can be calculated from equation (3.4) as

$$\begin{aligned} 2 \sin(\phi - \psi) \sin^2 \frac{\theta}{2} &= E_{21} + E_{12} \\ 2 \cos(\phi - \psi) \sin^2 \frac{\theta}{2} &= E_{11} - E_{22} \\ 2 \sin(\phi + \psi) \cos^2 \frac{\theta}{2} &= E_{21} - E_{12} \\ 2 \cos(\phi + \psi) \cos^2 \frac{\theta}{2} &= E_{11} + E_{22} \end{aligned} \quad (3.6)$$

This so called “gimbal lock” is the main disadvantage of the Euler angle scheme. It is possible for a large number of dynamic problems to avoid these positions by selecting an Euler angle scheme for the specific problem that would not result in this singularity.

### 3-1-3 Euler angle kinematics

The angular velocity in terms of the static base  $\vec{\underline{E}}_s$  for the 3-1-3 Euler angle parameters, follows from figure 4

$$\vec{\omega}^s = \dot{\phi} \vec{e}_3^s + \dot{\theta} \vec{\underline{E}}_\xi^s \cdot \vec{e}_1^\xi + \dot{\psi} \vec{\underline{E}}_\xi^s \cdot \vec{\underline{E}}_\eta^s \cdot \vec{e}_3^\eta \quad (3.7)$$

and in terms of the rotating base  $\vec{\underline{E}}_r$ , note that  $\vec{e}_3^s = \vec{e}_3^\xi$ ,  $\vec{e}_1^\xi = \vec{e}_1^\eta$  and  $\vec{e}_3^\eta = \vec{e}_3^r$

$$\vec{\omega}^r = \dot{\phi} \vec{\underline{E}}_\eta^r \cdot \vec{\underline{E}}_\xi^\eta \cdot \vec{e}_3^\xi + \dot{\theta} \vec{\underline{E}}_\eta^r \cdot \vec{e}_1^\eta + \dot{\psi} \vec{e}_3^r \quad (3.8)$$

The rotations are not about perpendicular axes and the resulting equations are therefore non-linear. In vector-matrix format

$$\vec{\omega}^r = \underline{\underline{\mathbf{B}}}_{313}^r \cdot \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}, \quad \vec{\omega}^s = \underline{\underline{\mathbf{B}}}_{313}^s \cdot \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (3.9)$$

with

$$\underline{\underline{\mathbf{B}}}_{313}^r = \begin{bmatrix} \sin \theta \sin \psi & \cos \psi & 0 \\ \sin \theta \cos \psi & -\sin \psi & 0 \\ \cos \theta & 0 & 1 \end{bmatrix} \quad (3.10a)$$

$$\underline{\underline{\mathbf{B}}}_{313}^s = \begin{bmatrix} 0 & \cos \phi & \sin \theta \sin \phi \\ 0 & \sin \phi & -\sin \theta \cos \phi \\ 1 & 0 & \cos \theta \end{bmatrix} = \underline{\underline{\mathbf{E}}}_{313}^s \cdot \underline{\underline{\mathbf{B}}}_{313}^r \quad (3.10b)$$

The time derivatives of the Euler angles are from the inverse of equation (3.10)

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \left[ \underline{\underline{\mathbf{B}}}_{313}^r \right]^{-1} \cdot \vec{\omega}^r = \left[ \underline{\underline{\mathbf{B}}}_{313}^s \right]^{-1} \cdot \vec{\omega}^s \quad (3.11)$$

with

$$\left[ \underline{\underline{\mathbf{B}}}_{313}^r \right]^{-1} = \frac{1}{\sin \theta} \begin{bmatrix} \sin \psi & \cos \psi & 0 \\ \sin \theta \cos \psi & -\sin \theta \sin \psi & 0 \\ -\cos \theta \sin \psi & -\cos \theta \cos \psi & \sin \theta \end{bmatrix} \quad (3.12a)$$

$$\left[ \underline{\underline{\mathbf{B}}}_{313}^s \right]^{-1} = \frac{1}{\sin \theta} \begin{bmatrix} -\cos \theta \sin \phi & \cos \theta \cos \phi & \sin \theta \\ \sin \theta \cos \phi & \sin \theta \sin \phi & 0 \\ \sin \phi & -\cos \phi & 0 \end{bmatrix} \quad (3.12b)$$

It is clear from equation (3.12) that the matrix becomes singular and the time derivatives of the Euler angles are undefined for  $\sin \theta = 0$ . This corresponds with the gimbal lock situation mentioned earlier and rules out a general numerical method to obtain the orientation of a base.



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