

# What to expect – an R vignette for `expectreg`

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## Abstract

`expectreg` is an R package for estimating expectile curves from univariate and multivariate data. Expectile curves are a valuable least squares alternative to quantile regression which is based on linear programming techniques. `expectreg` provides a number of functions for different approaches taken to estimate expectiles investigated since their introduction in [NEWBY and POWELL(1987)] using asymmetric least squares.

## 1 Overview

This section offers an overview over the functions implemented in `expectreg`. It assumes that the user already installed the package successfully.

```
> library(expectreg)
> help(package = "expectreg")
> data(package = "expectreg")
```

will give you a short overview about the available help files of the package as well as the data that will be provided with `expectreg`. The package includes the following functions:

<code>rb</code>	Creates bases for a regression based on covariates
<code>demq</code>	Density of a special distribution developed by Roger Koenker [KOENKER(1992)]
<code>ebeta</code>	Expectiles of the beta distribution
<code>eemq</code>	Expectiles of a special distribution developed by Roger Koenker
<code>enorm</code>	Expectiles of the normal distribution
<code>eunif</code>	Expectiles of the uniform distribution
<code>expectreg.boost</code>	Expectile regression using boosting
<code>expectreg.ls</code>	Expectiles regression of additive models
<code>expectreg.qp</code>	Expectile sheets with monotonicity constraints
<code>pemq</code>	Distribution function for a special distribution developed by Roger Koenker
<code>qemq</code>	Quantile function for a special distribution developed by Roger Koenker
<code>quant.boost</code>	Quantile regression using boosting
<code>remq</code>	Random variable generated from a special distribution developed by Roger Koenker

## 2 Expectiles in a nutshell

### 2.1 Introduction to expectiles using LAWS

Asymmetric least squares or least asymmetrically weighted squares (LAWS) is a weighted generalization of ordinary least squares (OLS) estimation. LAWS minimizes

$$S = \sum_{i=1}^n w_i(p)(y_i - \mu_i(p))^2,$$

with

$$w_i(p) = \begin{cases} p & \text{if } y_i > \mu_i(p) \\ 1 - p & \text{if } y_i \leq \mu_i(p) \end{cases}, \quad (1)$$

where  $y_i$  is the response and  $\mu_i(p)$  is the population expectile for different values of an asymmetry parameter  $p$  with  $0 < p < 1$ . The model is fitted by alternating between weighted regression and recomputing weights until convergence (when the weights do not change anymore). Equal weights ( $p = 0.5$ ) give a convenient starting point.

For the expectile curve  $\mu(p)$  several choices for the functional form are possible. The original proposal in [NEWHEY and POWELL(1987)] favored a linear model. We suggest a more flexible functional form for the expectile curve. [SCHNABEL and EILERS(2009)] proposed to model expectile curves with  $P$ -splines. Other types such as other splines, markov random field or other options are also possible (see [SOBOTKA and KNEIB(2010)]).

### 2.2 Expectile bundle model

In theory it is not possible that expectile curves cross, but in estimation practise it is often encountered due to sampling variation. The expectile bundle model is a location-scale type of model that allows for the simultaneous estimation of a set of expectiles. By its construction crossing over of curves is not possible. In the expectile bundle model the expectiles  $\mu(x, p)$  are defined by

$$\mu(x, p) = t(x) + c(p)s(x) \quad (2)$$

where  $t(x)$  is a common smooth trend of all expectile curves specified by a  $P$ -spline.  $c(p)$  is the asymmetry function of the bundle describing the spread, i.e. the set of standardized expectiles.  $s(x)$  represents the local width of the expectile bundle and is also formulated as a  $P$ -spline. The estimation procedure consists of two steps. In Step 1 the common trend  $t(x)$  is estimated. Then in step 2 we use the detrended response  $y - t(x)$  to estimate  $s(x)$  and  $c(p)$  in an iterative procedure.

The expectiles bundle model is explained in more detail in [SCHNABEL and EILERS(2010)].

### 2.3 Restricted regression quantiles

In [HE(1997)] proposed a version of restricted regression quantiles to avoid the crossing of quantile curves. His model for computing non-parametric conditional quantile functions takes the following form

$$y = f(x) + s(x)e.$$

[HE(1997)] takes a three-step procedure where he determines first the conditional median function and then in a second step estimate the smooth non-negative amplitude function. The third step consists of the step wise calculation of the “asymmetry factor”  $c_\alpha$  for each  $\alpha$ -quantile curve separately.

## 2.4 Expectile and quantile estimation using boosting

1. Initialize all model components as  $\hat{\mathbf{f}}_j^{[0]}(\mathbf{z}) \equiv \mathbf{0}$ ,  $j = 1, \dots, r$ . Set the iteration index to  $m = 1$ .
2. Compute the current negative gradient vector  $\mathbf{u}$  with elements

$$u_i = - \left. \frac{\partial}{\partial \eta} \rho(y_i, \eta) \right|_{\eta = \hat{\eta}^{[m-1]}(\mathbf{z}_i)}, \quad i = 1, \dots, n.$$

3. Choose the base-learner  $\mathbf{g}_{j^*}$  that minimizes the  $L_2$ -loss, i.e. the best-fitting function according to

$$j^* = \arg \min_{1 \leq j \leq r} \sum_{i=1}^n (u_i - \hat{g}_j(\mathbf{z})_i)^2$$

where  $\hat{\mathbf{g}}_j = \mathbf{S}_j \mathbf{u}$ .

4. Update the corresponding function estimate to  $\hat{\mathbf{f}}_{j^*}^{[m]} = \hat{\mathbf{f}}_{j^*}^{[m-1]} + \nu \hat{\mathbf{g}}_{j^*}$ , where  $\nu \in (0, 1]$  is a step size. For all remaining functions set  $\hat{\mathbf{f}}_j^{[m]} = \hat{\mathbf{f}}_j^{[m-1]}$ ,  $j \neq j^*$ .
5. Increase  $m$  by one. If  $m < m_{\text{stop}}$  go back to step 2., otherwise terminate the algorithm.

For expectile regression, the empirical risk is given the asymmetric least squares criterion (1) and the appropriate loss function is defined as  $\rho(y, \eta) = w(\tau)(y - \eta_\tau)^2$ . The corresponding negative gradient is therefore obtained as

$$u_i = 2w_i(\tau)(y_i - \eta_i).$$

## 3 Example and available data

Expectile estimation can be used in a almost any type of situation where one is interested in estimating smooth curves in non-central parts of the data under consideration. The data provided with the package are

```
> data(india)
> data(dutchboys)
```

`india` consists of a data sample of 4000 observations with 6 variables from a ‘Demographic and Health Survey’ about malnutrition of children in India. Data set only contains 1/10 of the observations and some basic variables to enable first analyses. Details are given in [FENSKE *et al.*(2009)].

`dutchboys` contains data from the Fourth Dutch growth study and includes 6848 observations on 10 variables. More information can be found in [VAN BUUREN and FREDRIKS(2001)].

### 3.1 Basic examples

The basic function `expectreg.ls` can be used to estimate 11 expectiles curves for different levels of asymmetry parameter  $p$ . The results are shown in the following graph.

```
> data(dutchboys)
> exp.l <- expectreg.ls(dutchboys[,3] ~ rb(dutchboys[,2], "pspline"), smooth="acv")
```

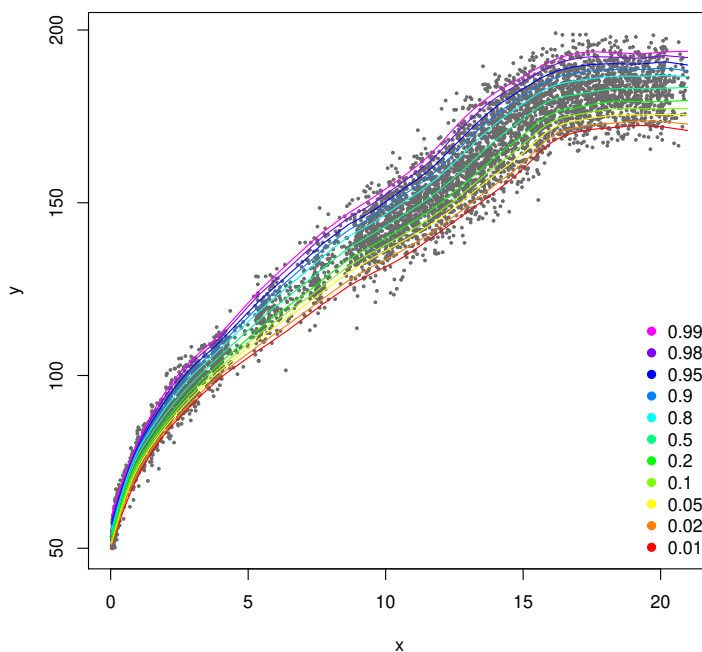


Figure 1: Expectile curves estimated using `expectreg.ls`

Due to the large number of observations in the data set crossing of curves is already unlikely to happen. Nevertheless we apply also the expectile bundle model implemented in `expectile.bundle` to this example.

```
> exp.b <- expectreg.ls(dutchboys[,3] ~ rb(dutchboys[,2], "pspline"), smooth="none", estimate
```

Additionally we analyze the data with the algorithm proposed in [HE(1997)] implemented in `expectile.restricted`.

```
> exp.r <- expectreg.ls(dutchboys[,3] ~ rb(dutchboys[,2], "pspline"), smooth="schall", estima
```

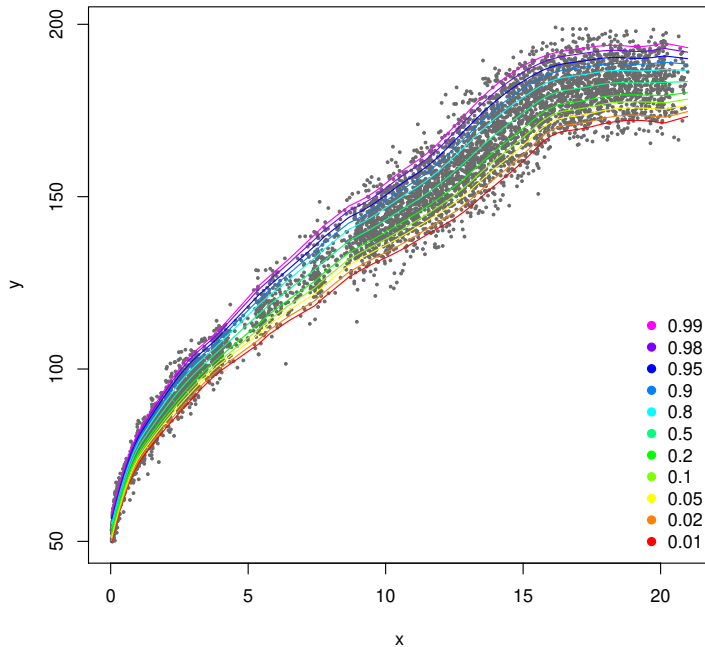


Figure 2: Expectile curves estimated using `expectreg.ls` with bundle estimate

### 3.2 Applied boosting

```
> exp.boost <- expectreg.boost(hgt ~ bbs(age,df=5,degree=2),dutchboys,mstop=rep(500,11))
```

## References

- [VAN BUUREN and FREDRIKS(2001)] VAN BUUREN, S., and A. M. FREDRIKS, 2001 Worm plot: A simple diagnostic device for modeling growth reference curves. *Statistics in Medicine* **20**: 1259–1277.
- [FENSKE *et al.*(2009)] FENSKE, N., T. KNEIB, and T. HOTHORN, 2009 Identifying risk factors for severe childhood malnutrition by boosting additive quantile regression. Technical Report 52, University of Munich.
- [HE(1997)] HE, X., 1997 Quantile curves without crossing. *The American Statistician* **51**: 186–192.
- [KOENKER(1992)] KOENKER, R., 1992 When are expectiles percentiles? (solution). *Economic Theory* **9**: 526–527.
- [NEWHEY and POWELL(1987)] NEWHEY, W. K., and J. L. POWELL, 1987 Asymmetric least squares estimation and testing. *Econometrica* **55**: 819–847.

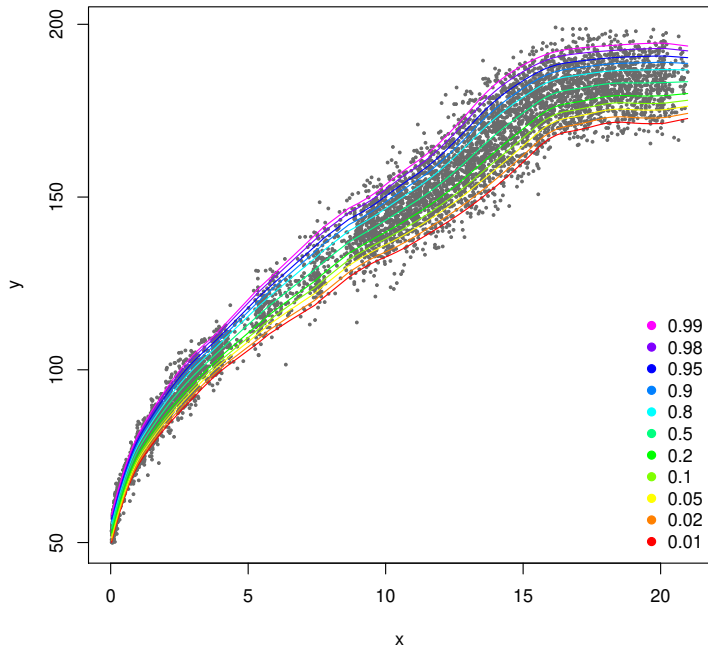


Figure 3: Expectile curves estimated using `expectreg.ls` with restricted estimate

[SCHNABEL and EILERS(2009)] SCHNABEL, S. K., and P. H. C. EILERS, 2009b  
Optimal expectile smoothing. *Computational Statistics and Data Analysis*  
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Graphical Statistics* (Submitted).

[SOBOTKA and KNEIB(2010)] SOBOTKA, F., and T. KNEIB, 2010 Geoadditive  
Expectile Regression. *Computational Statistics and Data Analysis*, doi:  
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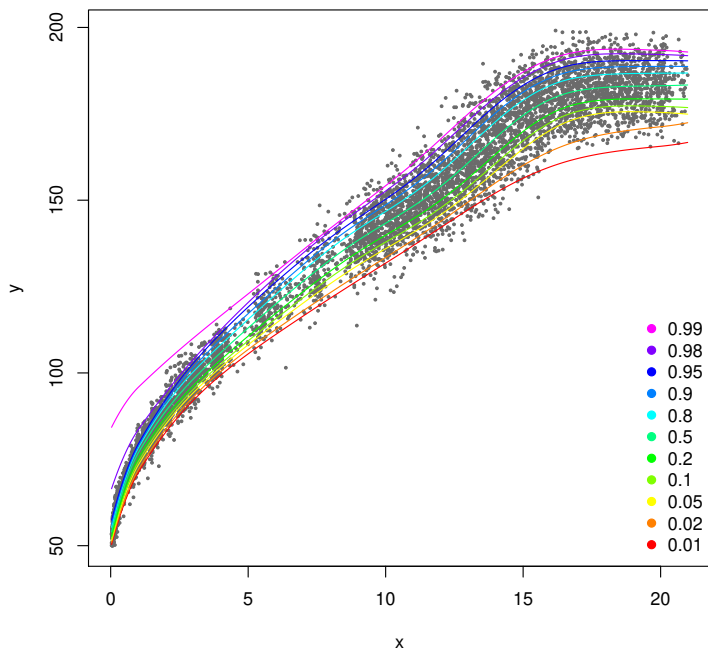


Figure 4: Expectile curves estimated using `expectreg.boost`